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Numerical Simulation of 3D Thermal-Fluid Coupled Model in Porous Medium

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Abstract

This article has presented finite difference schemes of 3D coupled model of seepage field and temperature field based on Darcy's seepage law and heat conduction rule in porous media, and gave the corresponding algorithm and main codes of the numerical computation. Numerical results showed that the numerical method of 3D coupled model is very efficient.

Keywords: 3D Thermal-Fluid Coupled Model; Difference Scheme; Numerical Simulation; Porous Medium

1 INTRODUCTION

The basic law of flow in porous medium was firstly presented by Mr. Darcy who was a famous French hydraulics in the middle of the 19th century. At present, the numerical methods of the hydrogeology and heat transfer models in porous media and rock matrix-fractured media are popular concentrated [1, 2, 3, 4, 7, 9]. The numerical simulations methods for these 1D and 2D models have made great progress, respectively [4, 8, 9]. And numerical methods for coupled model such as Burgers' equations have been presented in [10, 11, 12]. However, the numerical simulation practice of 3D thermal-fluid coupled model in porous medium still is difficult to be solved. In this paper, finite difference scheme of 3D thermal-fluid coupled model and compare the numerical results have been presented with the exact solution by the numerical example.

2 THREE DIMENSIONAL THERMAL-FLUID COUPLED MODEL

Considering the problem that fluids flow in porous media, the factors that cause the change of the water head include heat convection, heat transfer effect, mechanical dispersion, heat exchange and so on[2].

By the principle of conservation of energy, we have

$$\frac{\partial (c_w T_w n)}{\partial t} = \left[\frac{\partial}{\partial x}\left(K_{wb} n \frac{\partial T_w}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_{wb} n \frac{\partial T_w}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_{wb} n \frac{\partial T_w}{\partial z}\right)\right] - \left[\frac{\partial}{\partial x}(c_w V T_w)\right] + K_{sw}(T_s - T_w). \quad (1)$$

The symbols n, c_w, T_s and T_w denote the porosity of media, specific heat of water, the solidus temperature and the water temperature, respectively. The symbol K_{wb} denotes hydrodynamic dispersion coefficient, and K_{sw} denotes heat transfer coefficient between solid phase and the aqueous phase.

And based on the Darcy's law [1], we can get

$$V = -K\left(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}, \frac{\partial H}{\partial z}\right), \quad (2)$$

Where $V(x,y,z,t) = u(x,y,z,t)i + v(x,y,z,t)j + w(x,y,z,t)k$ represents the seepage velocity, K is the permeability coefficient of porous media, H represents the water head value.

Seepage continuity equation can be expressed as

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{\partial^2 H}{\partial z^2} = 0. \quad (3)$$

Since the analytical solution of equations (1)-(3) is usually difficult to seek out, its approximate solution is found out by finite difference method.

3 Difference Schemes of the Heat Transfer and Continuity Equations

Since equation (1) is very difficult to calculate, some assumptions are made to simple the equation. It is assumed that the porosity n is a const, the specific heat c_w of water is a constant, and the solidus temperature T_s is equal to the water temperature T_w. Then the above equation (1) can be simplified as the following equation

$$\frac{\partial T_w}{\partial t} = \frac{K_{wb}}{c_w}\left[\frac{\partial}{\partial x}\left(\frac{\partial T_w}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial T_w}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial T_w}{\partial z}\right)\right]-\frac{1}{n}\left[\frac{\partial}{\partial x}(uT_w)+\frac{\partial}{\partial x}(vT_w)+\frac{\partial}{\partial x}(wT_w)\right]. \quad (4)$$

Omitting all the subscripts, the equation (4) is expressed as

$$\frac{\partial T}{\partial t} = \frac{k1}{c}\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right]-\frac{1}{n}\left[u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right]. \quad (5)$$

Difference scheme of equation (5) is

$$\frac{T_{i,j}^{k,t+1}-T_{i,j}^{k,t}}{\tau} = \frac{k1}{c}\left[\frac{T_{i+1,j}^{k,t}-2T_{i,j}^{k,t}+T_{i-1,j}^{k,t}}{h_1^2}+\frac{T_{i,j+1}^{k,t}-2T_{i,j}^{k,t}+T_{i,j-1}^{k,t}}{h_2^2}+\frac{T_{i,j}^{k+1,t}-2T_{i,j}^{k,t}+T_{i,j}^{k-1,t}}{h_3^2}\right]-\frac{1}{n}\left[u_{i,j}^k\frac{T_{i+1,j}^{k,t}-T_{i-1,j}^{k,t}}{2h_1}+v_{i,j}^k\frac{T_{i,j+1}^{k,t}-T_{i,j-1}^{k,t}}{2h_2}+w_{i,j}^k\frac{T_{i,j}^{k+1,t}-T_{i,j}^{k-1,t}}{2h_3}\right]. \quad (6)$$

In the formula (6), set $k2 = \frac{k1}{c}$, $h_1 = h_2 = h_3 = h$. Then the formula (6) can be simplified to

$$\tau\left[\left(\frac{k2}{h^2}+\frac{w_{i,j}^k}{2nh}\right)T_{i,j}^{k-1,t}+\left(\frac{k2}{h^2}+\frac{u_{i,j}^k}{2nh}\right)T_{i-1,j}^{k,t}+\left(\frac{k2}{h^2}+\frac{v_{i,j}^k}{2nh}\right)T_{i,j-1}^{k,t}+\left(\frac{k2}{h^2}-\frac{v_{i,j}^k}{2nh}\right)T_{i,j+1}^{k,t}+\left(\frac{k2}{h^2}-\frac{u_{i,j}^k}{2nh}\right)T_{i+1,j}^{k,t}+\left(\frac{k2}{h^2}-\frac{w_{i,j}^k}{2nh}\right)T_{i,j}^{k+1,t}\right]+\left(\frac{1}{\tau}-6\frac{k2*\tau}{h^2}\right)T_{i,j}^{k,t}-T_{i,j}^{k,t+1}=0 \quad (7)$$

Using the centered-difference formula, the equation (3) at the points (x_i, y_j, z_k) can be expressed as

$$\frac{H_{i+1,j}^k-2H_{i,j}^k+H_{i-1,j}^k}{h_1^2}+\frac{H_{i,j+1}^k-2H_{i,j}^k+H_{i,j-1}^k}{h_2^2}+\frac{H_{i,j}^{k+1}-2H_{i,j}^k+H_{i,j}^{k-1}}{h_3^2}=0. \quad (8)$$

Set $h_1 = h_2 = h_3 = h$, the formula (8) can be simplified as

$$H_{i,j}^{k+1}+H_{i+1,j}^k+H_{i,j+1}^k-6H_{i,j}^k+H_{i,j}^{k-1}+H_{i+1,j}^k+H_{i,j-1}^k=0 \quad (9)$$

Where $H_{i,j}^{k-1}$ are the water head values at mesh points (x_i, y_j, z_k).

And then the Dirichlet boundary condition $y = f(x, y, z)$ is considered. The function value at discrete mesh points on boundary can be given firstly. Then combined with formula (8), all water head function values at the mesh points can be calculated.

When the water head values at discrete points are known, according to equation (2) and (8), we can calculate seepage velocity at each mesh grid. The discrete forms of seepage velocity V are shown as

$$u_{i,j}^k = -K\frac{H_{i+1,j}^k-H_{i-1,j}^k}{2h},\ v_{i,j}^k = -K\frac{H_{i,j+1}^k-H_{i,j-1}^k}{2h},\ w_{i,j}^k = -K\frac{H_{i,j}^{k+1}-H_{i,j}^{k-1}}{2h}. \quad (10)$$

At the boundary points, the following formulas can be obtained

$$u_{i,j}^k = -K\frac{H_{i+1,j}^k-H_{i,j}^k}{h}, v_{i,j}^k = -K\frac{H_{i,j+1}^k-H_{i,j}^k}{h}, w_{i,j}^k = -K\frac{H_{i,j}^{k+1}-H_{i,j}^k}{h} \quad (11)$$

If the boundary conditions and initial conditions are known in the given domain, the numerical approximation of unknown function values at discrete points can be calculated by the formulas (6), (8) (10) and (11).

4 Algorithm and the Numerical Example

From the difference schemes of the above equations, we give the following algorithm and main codes.

Input n0;
Set h=1/n0; n=n0+1;
A=eye(n*n*n);b=zeros(n*n*n,1);x0=b;x1=b;

```
for k=1,…,n
        for i=1,…,n
            for j=1,…,n
                set m=(k-1)*n^2+(i-1)*n+j;
                  if k>=2&&k<n&&i>=2&&i<n&&j>=2&&j<n;
                        A(m,m)=-6;A(m,m-1)=1;A(m,m+1)=1;
                        A(m,m-n)=1;A(m,m+n)=1;A(m,m-n^2)=1;A(m,m+n^2)=1;
                  end
                  if k==1
                    b(m)=0;
                  elseif k==n
                     b(m)=H((i-1)*h,(j-1)*h,1);
                  end
                  if i==1
                    b(m)=H(0,(j-1)*h,(k-1)*h);
                  elseif i==n
                     b(m)=H(1,(j-1)*h,(k-1)*h);
                  end
                 if j==1
                    b(m)=H((i-1)*h,0,(k-1)*h);
                 elseif j==n
                     b(m)=H((i-1)*h,1,(k-1)*h);
                 end
              end
         end
end
x=A\b;
clear i j k m

For k=1,…, n
      for i=1,…, n
            for j=1,…, n
                set m=(k-1)*n^2+(i-1)*n+j;
                  x0(m)=H((i-1)*h,(j-1)*h,(k-1)*h);
                  x1(m)=H0((i-1)*h,(j-1)*h,(k-1)*h);
            end
      end
end
Output x0; x;
Set B=zeros(n^3,3); B0=B; B0(:,1)=x0; B0(:,2)=x0; B0(:,3)=x1; B1=B0;
for k=1,…,n
        for i=1,…,:n
            for j=1,…,:n
                m=(k-1)*n^2+(i-1)*n+j;
                  if k>=2&&k<n&&i>=2&&i<n&&j>=2&&j<n
                        B(m,1)=(x(m+n)-x(m-n))/(2*h); B(m,2)=(x(m+1)-x(m-1))/(2*h);
                        B(m,3)=(x(m+n^2)-x(m-n^2))/(2*h);B1(m,1)=0;B1(m,2)=0;
                              B1(m,3)=0;
                  end
             end
```

```
        end
end
B1;
  B2=B1+B; B2-B0;B=B2;
Set K=1; % K is the permeability
Set C=-K*B;C0=-K*B0;    % C,C0 are the velocity at discrete points
clear i j k m
Input K1    t0
Set N=0.3; % Porosity
Set n2=n^3*((t0*n0)+1);    % n2 is the number of the discrete points
Set E=zeros(n2,1); T1=E;
For t=1,…, (t0*n0)+1
      for k=1,…, n
            for i=1,…, n
                  for j=1,…, n
                     set m=(t-1)*n*n*n+(k-1)*n*n+(i-1)*n+j;
                        T1(m)=T0((i-1)*h,(j-1)*h,(k-1)*h,(t-1)*h/t0);
                        n3=(k-1)*n^2+(i-1)*n+j;
                  if t==1
                        then set E(m)=T0((i-1)*h,(j-1)*h,(k-1)*h,(t-1)*h/t0);
                  end
            if t>1&&k>=2&&k<n&&i>=2&&i<n&&j>=2&&j<n
Then set
E(m)=(K1/(h*t0)+C(n3,3)/(2*t0*N))*E(m-n^2-n^3)+E(m+n^2-n^3)*(K1/(h*t0)-C(n3,3)/(2*t0*N))+E(m-n-n^3)*(K
1/(h*t0)+C(n3,1)/(2*t0*N))+E(m+n-n^3)*(K1/(h*t0)-C(n3,1)/(2*t0*N))+E(-n^3+m-1)*(K1/(h*t0)+C(n3,2)/(2*t0*N
))+E(m+1-n^3)*(K1/(h*t0)-C(n3,2)/(2*t0*N))+E(m-n^3)*(1-6*K1/(h*t0));
                  end
                     if k==1
                        set E(m)=T0((i-1)*h,(j-1)*h,(k-1)*h,(t-1)*h/t0);
                           elseif k==n
                                 then set E(m)=T0((i-1)*h,(j-1)*h,(k-1)*h,(t-1)*h/t0);
                           end
                     if i==1
                        set E(m)=T0((i-1)*h,(j-1)*h,(k-1)*h,(t-1)*h/t0);
                           elseif i==n
                                 then set    E(m)=T0((i-1)*h,(j-1)*h,(k-1)*h,(t-1)*h/t0);
                           end
                     if j==1
                        set E(m)=T0((i-1)*h,(j-1)*h,(k-1)*h,(t-1)*h/t0);
                           elseif j==n
                                 then set E(m)=T0((i-1)*h,(j-1)*h,(k-1)*h,(t-1)*h/t0);
                           end
                     end
               end
        end
end
OUTPUT
```

Then, based on the above algorithm, we solve the following 3D coupled model consisting of equations (12) and (13).

$$\begin{cases} \frac{\partial^2 H}{\partial x^2}+\frac{\partial^2 H}{\partial y^2}+\frac{\partial^2 H}{\partial z^2}=0, \quad 0<x,y,z<1 \\ H\big|_{x=0}=e^{y}\sin\left(\sqrt{2}z\right), \ 0\le y,z\le 1 \\ H\big|_{y=0}=e^{x}\sin\left(\sqrt{2}z\right), \ 0\le x,z\le 1 \\ H\big|_{z=0}=0, \ 0\le x,y\le 1 \\ H\big|_{x=1}=e^{1+y}\sin\left(\sqrt{2}z\right), \ 0\le y,z\le 1 \\ H\big|_{y=1}=e^{x+1}\sin\left(\sqrt{2}z\right), \ 0\le x,z\le 1 \\ H\big|_{z=1}=e^{x+y}\sin\left(\sqrt{2}\right), \ 0\le x,y\le 1 \end{cases} \tag{12}$$

$$\begin{cases} \frac{\partial T}{\partial t}=\frac{2}{5}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)-\frac{10}{3}\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right), \quad 0<x,y,z<1, \quad 0<t<1 \\ T\big|_{t=0}=\sqrt{2}z, \ 0\le x,y,z\le 1 \\ T\big|_{x=0}=\frac{20te^{y}\cos\left(\sqrt{2}z\right)}{3}+\sqrt{2}z, \quad 0\le y,z\le 1, \ 0\le t\le 1 \\ T\big|_{y=0}=\frac{20te^{x}\cos\left(\sqrt{2}z\right)}{3}+\sqrt{2}z, \quad 0\le x,z\le 1, \ 0\le t\le 1 \\ T\big|_{z=0}=\frac{20te^{x+y}}{3}, \ 0\le x,y\le 1, \ 0\le t\le 1 \\ T\big|_{x=1}=\frac{20te^{1+y+z}\cos\left(\sqrt{2}z\right)}{3}+\sqrt{2}z, 0\le y,z\le 1, 0\le t\le 1 \\ T\big|_{y=1}=\frac{20te^{x+1+z}\cos\left(\sqrt{2}z\right)}{3}+\sqrt{2}z, 0\le x,z\le 1, \ 0\le t\le 1 \\ T\big|_{z=1}=\frac{20te^{x+y+1}\cos\left(\sqrt{2}\right)}{3}+\sqrt{2}, \quad 0\le x,y\le 1, \ 0\le t\le 1 \end{cases} \tag{13}$$

Where $(u,v,w)=-\left(\frac{\partial H}{\partial x},\frac{\partial H}{\partial y},\frac{\partial H}{\partial z}\right)$.

The analytical solution to equation (11) is $H(x,y,z)=e^{x+y}\sin\left(\sqrt{2}z\right)$, and the computed analytical solution to equation (13) is $T(x,y,z,t)=\frac{20te^{x+y}\cos\left(\sqrt{2}z\right)}{3}+\sqrt{2}z$.

In this example, the definite domain is given as the unit cube. The cube is equidistant subdivision, each side length is 12 equal parts, the time period is unit time and time step is 1/1800.

The numerical results of the temperature T and water head value H are shown as follows.

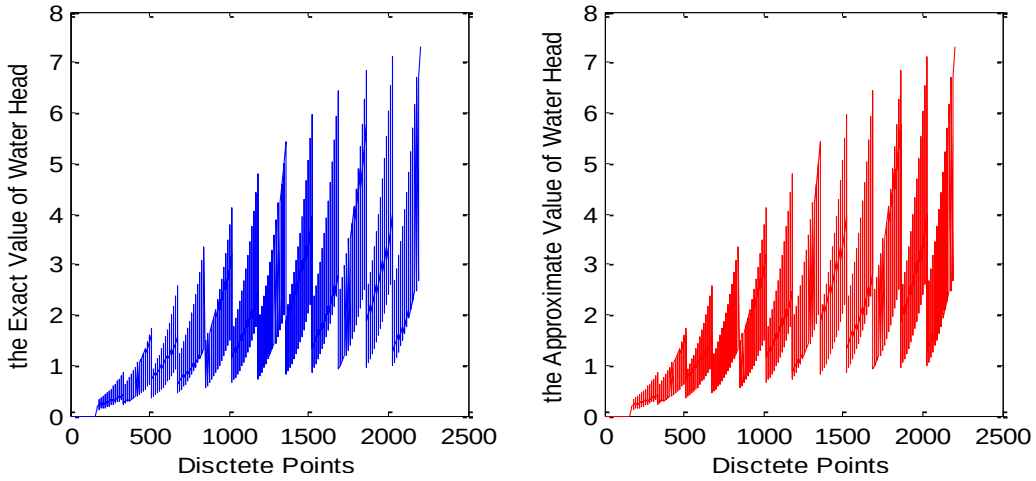


FIG. 1 COMPARISON OF THE EXACT AND APPROXIMATION VALUE OF THE WATER HEAD

TABLE 1 COMPARISON OF THE EXACT AND APPROXIMATION VALUE OF H(x,y,z) WITH x=1/3, y=1/6, h=1/12

z	**The exact value** H	**The approximate value** H1	**Error** H1-H
0.083333333333333	0.193854200372343	0.193883391857379	0.000029191485036
0.166666666666667	0.385019096078945	0.385076676916620	0.000057580837675
0.250000000000000	0.570842690696917	0.570927018974783	0.000084328277866
0.333333333333333	0.748747086718487	0.748855600447801	0.000108513729314
0.416666666666667	0.916264248262269	0.916393331735950	0.000129083473680
0.500000000000000	1.071070239693822	1.071215018405726	0.000144778711904
0.583333333333333	1.211017465169250	1.211171500284468	0.000154035115218
0.666666666666667	1.334164461848409	1.334319297162373	0.000154835313965
0.750000000000000	1.438802833461809	1.438947315539083	0.000144482077274
0.833333333333333	1.523480950586641	1.523600184442877	0.000119233856236
0.916666666666667	1.587024088842251	1.587097798339812	0.000073709497560

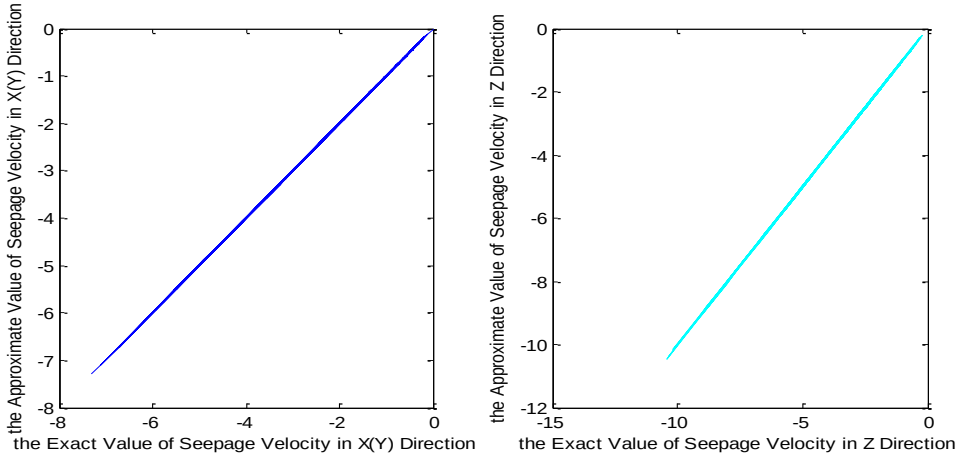


FIG. 2 COMPARISON OF THE EXACT AND APPROXIMATION VALUE OF SEEPAGE VELOCITY

TABLE 2 COMPARISON OF THE EXACT AND SOME APPROXIMATION VALUE OF T(x,y,z)
WITH x=1/3, y=1/6, z=1/4,h=1/12, τ=1/1800, k2=0.01

Time t	**The exact value** T	**The approximate value** T1	**Error** T1-T
0.0006	0.3593	0.3593	-0.0000
0.0172	0.5311	0.5308	-0.0003
0.0339	0.7030	0.7024	-0.0006
0.0506	0.8749	0.8740	-0.0008
0.0672	1.0467	1.0458	-0.0009
0.0839	1.2186	1.2181	-0.0005
0.1006	1.3904	1.3910	0.0006
0.1172	1.5623	1.5645	0.0022
0.1672	2.0779	2.0854	0.0075
0.1839	2.2497	2.2590	0.0093
0.2006	2.4216	2.4324	0.0108
0.3506	3.9684	3.9936	0.0252
0.3672	4.1402	4.1671	0.0269
0.3839	4.3121	4.3407	0.0286
0.4006	4.4839	4.5142	0.0303
0.4172	4.6558	4.6877	0.0319
0.4339	4.8277	4.8612	0.0336
0.5506	6.0307	6.0762	0.0455
0.6672	7.2337	7.2911	0.0574
0.6839	7.4056	7.4646	0.0590
0.7006	7.5774	7.6382	0.0607
0.7172	7.7493	7.8120	0.0627
0.8006	8.6086	8.6688	0.0602
0.8172	8.7804	8.8194	0.0390

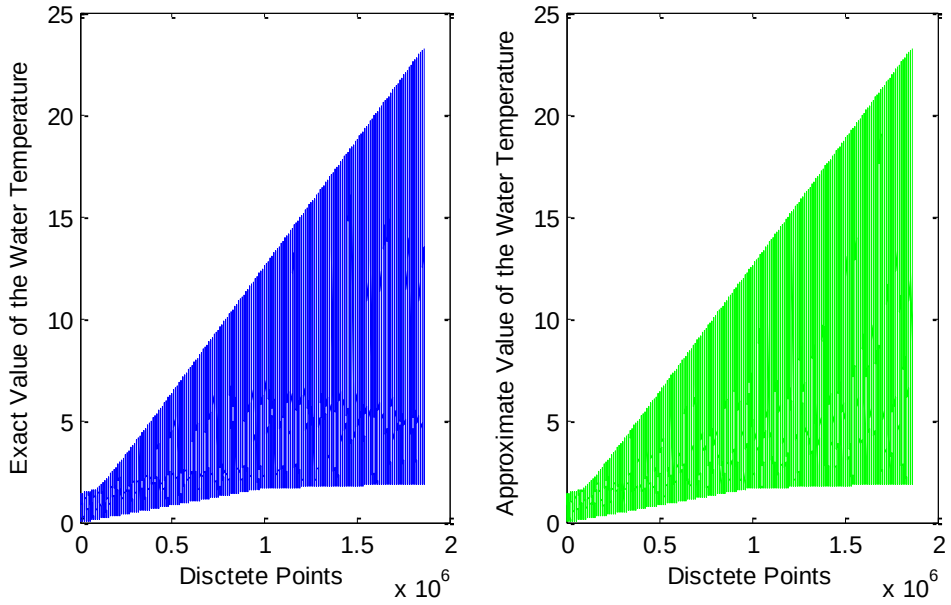


FIG. 3 COMPARISON OF THE EXACT AND APPROXIMATION VALUE OF WATER TEMPERATURE

5 CONCLUSIONS

This paper presented the difference scheme of the three-dimensional thermal-fluid coupled mathematic model and tested the given algorithm with the numerical example which has a large amount of calculation. The numerical simulation example was given in the special definite domain and the results of the compute test showed the efficiency of the algorithm. The numerical method was taken into consideration that can reduce the compute cost and the numerical simulation examples defined in the general domain.

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