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Novel Exponential Convergence of Networks on Time Scales

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Abstract

In this paper, we analyze exponential convergence of neural networks based on basic properties of general exponential function on time scales. By employing the time scales calculus theory and Lyapunov functional method, we establish sufficient conditions to ensure exponential convergence of networks. Our stability results on time scales can include corresponding continuous-time networks and discrete-time ones.

Keywords: Neural Networks; On Time Scales; Exponential Convergence.

1 INTRODUCTION

It is well known that a variety of models for neural networks have been successfully applied to signal processing, image processing, associative memories, patter classification and optimization [1-2]. For applications of neural networks, there is a shared requirement of raising the networks convergence speed in order to cut down the time of neural computing, so it is meaningful to study the exponential convergence and exponential stability of neural networks. Some relative results could be found in [4-5]. It is well known that stability results of discrete-time networks are important and it is troublesome to study the exponential convergence and exponential stability for continuous and discrete models, respectively. So it is significant to study networks on time scales which can unify the continuous and discrete situations. For the theory of time scales, we can refer to Stefan Hilger's original works in [6]. The books by Bohner and Peterson [7-8] also summarize and organize much of time scales calculus. What is more, several authors have expounded on various aspects of this theory [9-11].

Consider the neural networks described by the following dynamical equation on time scale $\mathbb{T}$:

$$x_i^{\Delta}(t) = -d_i x_i(t) + \sum_{j=1}^{n} T_{ij} g_j(x_j(t)) + I_i,\ i \in \mathcal{N} := \{1,2,\cdots n\} \tag{1.1}$$

where $x_i(t): \mathbb{T} \to \mathbb{R}$ and $(x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$, $d_i > 0$. $(T_{ij})_{n\times n}$ is a constant matrix and $g_i(s): \tilde{\mathbb{T}} \to \mathbb{R}$ are the neural input-output activations for all $s \in \tilde{\mathbb{T}}$. Throughout this paper, we always assume that $\mathbb{T}_0 = \mathbb{T}^k \cap [0, +\infty]$.

2 SOME PRELIMINARIES ON TIME SCALES

Definition 1 A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. The forward jump operator σ on $\mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ for all $t \in \mathbb{T}$. The backward jump operator ρ on $\mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ for all $t \in \mathbb{T}$. The $\mathbb{T}^k$ is derived form $\mathbb{T}$ as follows: if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. Then $(\mathbb{T}, \sigma, \rho)$ is called a time scale.

Definition 2 For a function $f: \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^k$, the delta derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

if f is continuous at t and t is right-scattered. If t is not right-scattered then the derivative is defined by

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(\sigma(t))-f(t)}{\sigma(t)-t}=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

provided this limit exists.

Definition 3 [7] Let $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}^k$, $f^{\Delta}(t)$ is the delta derivative of f, $D^{+}f^{\Delta}(t)$ is said to be Dini derivative of $f(t)$, if given any $\varepsilon>0$, there exists a right neighbour $U_{\varepsilon}\subseteq U$ of t such that

$$\frac{f(\sigma(t))-f(s)}{\sigma(t)-s}<D^{+}f^{\Delta}(t)+\varepsilon,\qquad \forall s\in U_{\varepsilon},\quad s>t.$$

A function $F:\mathbb{T}\to\mathbb{R}$ is called an anti-derivative of $f:\mathbb{T}\to\mathbb{R}$ provided $F^{\Delta}(t)=f(t)$ holds for all $t\in\mathbb{T}^k$. Definite the Cauchy integral of f by $\int_a^b f(t)\Delta t=F(b)-F(a)$ for $a,b\in\mathbb{T}^k$. $p:\mathbb{T}\to\mathbb{R}$ is said to be regressive provided $1+\mu(t)p(t)\neq 0$ for all $t\in\mathbb{T}^k$. Denote by $\mathcal{R}$ the set of all regressive and rd-continuous functions. Define the set of all positively regressive functions by $\mathcal{R}^{+}=\{p\in\mathcal{R}:1+\mu(t)p(t)>0\}$ for all $t\in\mathbb{T}^k$.

Definition 4 For $h>0$, we define the cylinder transformation $\xi_h:\mathbb{C}_h\to\mathbb{Z}_h$ by $\xi_h(z)=\frac{1}{h}Log(1+zh)$, where Log is the principal logarithm function. If $p\in\mathcal{R}$, then we define the exponential function by

$$e_p(t,s)=\exp\left(\int_s^t\xi_{\mu(\tau)}(p(\tau))\Delta\tau\right)\ \text{for}\ s,t\in\mathbb{T}^k.$$

Definition 5 If $p,q\in\mathcal{R}$, we define the circle plus $\oplus$ addition by $(p\oplus q)(t):=p(t)+q(t)+\mu(t)p(t)q(t)$ for all $t\in\mathbb{T}^k$. And the function $\ominus p$ is defined by $(\ominus p)(t):=-\frac{p(t)}{1+\mu(t)p(t)}$ for all $t\in\mathbb{T}^k$.

Lemma 1 If $p\in\mathcal{R}$ and fix $t_0\in\mathbb{T}^k$, then the exponential function $e_p(\cdot,t_0)$ is the unique solution of the initial value problem $x^{\Delta}=p(t)x$, $x(t_0)=1$ and

$$(i)\ e_p(\sigma(t),s)=(1+\mu(t)p(t))e_p(t,s);\ (ii)\ e_p(t,s)e_q(t,s)=e_{p\oplus q}(t,s).$$

The following is the comparison theorem in [7].

Lemma 2 Let $y,f\in C_{rd}$ and $p\in\mathcal{R}^{+}$, then $y^{\Delta}(t)\le p(t)y(t)+f(t)$ implies

$$y(t)\le y(t_0)e_p(t,t_0)+\int_{t_0}^t e_p(t,\sigma(\tau))f(\tau)\Delta\tau,\ \text{for all}\ t\in\mathbb{T}^k.$$

3 EXPONENTIAL CONVERGENCE

Throughout this paper we assume that $g_i(\cdot)$ is strictly monotonous and there exist some constants $k_i>0$ $(i\in\mathcal{N})$ such that for $\forall u,v\in\tilde{\mathbb{T}}^k$

$$(A_1):\qquad 0\le\frac{g_i^{\tilde{\Delta}(u)}}{2k_i}\left(1+\frac{g_i(\tilde{\sigma}(u))-g_i(v)}{g_i(u)-g_i(v)}\right)\le 1,\quad u\neq v.$$

Lemma 3 Assume that (A_1) holds. For all $u,v\in\tilde{\mathbb{T}}^k$, we have

$$\int_v^u[g_i(s)-g_i(v)]\tilde{\Delta}s\ge\frac{1}{2k_i}[g_i(u)-g_i(v)]^2,\qquad (i\in\mathcal{N}).$$

***Proof*:** Define auxiliary functions as follows.

$$E_i(u)=\int_v^u[g_i(s)-g_i(v)]\tilde{\Delta}s-\frac{1}{2k_i}[g_i(u)-g_i(v)]^2,\qquad (i\in\mathcal{N}).$$

Then we get

$$E_i^{\tilde{\Delta}}(u)=[g_i(u)-g_i(v)]\left[1-\frac{g_i^{\tilde{\Delta}}(u)}{2k_i}\left(1+\frac{g_i(\tilde{\sigma}(u))-g_i(v)}{g_i(u)-g_i(v)}\right)\right] \qquad (u\neq v).$$

It follows from (A_1) that $E_i^{\tilde{\Delta}}(u)\geq 0$ if $u>v$; $E_i^{\tilde{\Delta}}(u)=0$ if $u=v$; $E_i^{\tilde{\Delta}}(u)\leq 0$ if $u<v$.

Hence $u=v$ is the minimum point of $E_i(u)$, so $E_i(u)\geq E_i(v)$ for any $u\in\tilde{\mathbb{T}}^k$. Therefore, we have

$$\int_v^u [g_i(s)-g_i(v)]\tilde{\Delta}s\geq\frac{1}{2k_i}[g_i(u)-g_i(v)]^2, \qquad (i\in\mathcal{N}).$$

Lemma 4 Assume that $\gamma>0$ and $-\gamma\in\mathcal{R}^+$, $t_0\in\mathbb{T}^k$ is fixed. Then there exists a $\delta\in(0,\gamma)$ with $\delta>0$ and $-\delta\in\mathcal{R}^+$ such that $[e_{-\gamma}(t,t_0)]^{\frac{1}{2}}=e_{-\delta}(t,t_0)$ for all $t,t_0\in\mathbb{T}^k$.

***Proof*:** Let $[e_{-\gamma}(t,t_0)]^{\frac{1}{2}}=e_{-\delta}(t,t_0)$, so we have $e_{-\gamma}(t,t_0)=e_{-\delta}(t,t_0)e_{-\delta}(t,t_0)=e_{(-\delta)\oplus(-\delta)}(t,t_0)$, i.e. $-\gamma=(-\delta)\oplus(-\delta)$. It follows form Definition 5 that $\gamma-\delta=\delta[1-\mu(\tau)\delta]$ $(\tau\in(t_0,t))$ which leads to $2\delta=\gamma+\mu(\tau)\delta^2$, then we have two cases to discuss.

- Case 1: $\mu(\tau)=0$. We have $\delta=\frac{\gamma}{2}$. Obviously, $\delta\in(0,\gamma)$.
- Case 1: $\mu(\tau)>0$. We have $0<\delta<\frac{\gamma}{2}$, so $\delta\in(0,\gamma)$ holds.

It follows from Case 1, Case 2 and $-\gamma\in\mathcal{R}^+$ that $-\delta\in\mathcal{R}^+$. This completes the proof.

Lemma 5 [3] If there exist constants $\alpha_i>0$ $(i\in\mathcal{N})$ such that the matrix

$$\left(-\frac{\alpha_i d_i}{k_i}\delta_{ij}+\frac{\alpha_i T_{ij}+\alpha_j T_{ji}}{2}\right)_{n\times n}, \qquad \delta_{ij}=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}$$

is negative definite, then (1.1) has a unique equilibrium point.

Theorem 1 If there exists a $\eta\in(0,\min_{1\leq i\leq n}\{d_i\})$ such that the matrix

$$(A_2):\quad\begin{cases}T_\eta=\left(-\frac{d_i-\eta}{k_i}\delta_{ij}+\frac{T_{ij}+T_{ji}}{2}\right)_{n\times n}, & \delta_{ij}=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}\\ 0\leq\mu^2(\tau)d_i^2\leq 1, \quad \tau\in\mathbb{T}_0\end{cases}$$

is negative semi-definite, then (1.1) has an unique equilibrium $(x_1^*,\cdots,x_n^*)^T$ which is globally exponential stable.

***Proof*:** The matrix T_η is negative semi-definite implies that the matrix

$$\left(-\frac{d_i}{k_i}\delta_{ij}+\frac{T_{ij}+T_{ji}}{2}\right)_{n\times n}$$

is negative definite, by Lemma 5, (1.1) has an unique equilibrium point $(x_1^*,\cdots,x_n^*)^T$. Next we definite a function $V(t)=\sum_{i=1}^n v_i(t)$ where $v_i(t)=\int_{x_i^*}^{x_i(t)}[g_i(\theta)-g_i(x_i^*)]\tilde{\Delta}\theta$, $(i\in\mathcal{N})$. It follows from (1.1) that

$$[x_i(t)-x_i^*]^{\Delta}=-d_i[x_i(t)-x_i^*]+\sum_{j=1}^n T_{ij}[g_j(x_j(t))-g_j(x_j^*)], \qquad t\in\mathbb{T}_0. \tag{3.1}$$

For delta derivative of $v_i(t)$, we have two cases to discuss.

Case 1: t is right-scatter, i.e. $\mu(t)>0$.

(i) If $x_i(t)=x_i(\sigma(t))$, then $x_i^{\Delta}(t)=0$. From (1.1) we have $x_i(t)=x_i^*$ and $v_i^{\Delta}(t)=0$ $(i\in\mathcal{N})$.

(*ii*) If $x_i(t) \neq x_i(\sigma(t))$, due to $\int_t^{\sigma(t)} f(s)\Delta s = f(t)\mu(t)$, then we have

$$v_i^{\Delta}(t) = \frac{\int_{x_i^*}^{x_i(\sigma(t))}[g_i(\theta) - g_i(x_i^*)]\tilde{\Delta}\theta - \int_{x_i^*}^{x_i(t)}[g_i(\theta) - g_i(x_i^*)]\tilde{\Delta}\theta}{\sigma(t) - t}$$
$$= \frac{\int_{x_i(t)}^{x_i(\sigma(t))}[g_i(\theta) - g_i(x_i^*)]\tilde{\Delta}\theta}{\sigma(t) - t} = [g_i(x_i(t)) - g_i(x_i^*)]x_i^{\Delta}(t)$$

Case 2: t is right-dense, i.e., $\mu(t) = 0$. It is obvious that $v_i^{\Delta}(t) = v_i'(t)$ and we get

$$v_i^{\Delta}(t) = v_i'(t) = [g_i(x_i(t)) - g_i(x_i^*)]x_i'(t) = [g_i(x_i(t)) - g_i(x_i^*)]x_i^{\Delta}(t).$$

Together with Case 1 and Case 2, we have

$$V^{\Delta}(t) \leq -\eta\sum_{i=1}^{n}[g_i(x_i(t)) - g_i(x_i^*)][x_i(t) - x_i^*]$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(-\frac{d_i - \eta}{k_i} + \frac{T_{ij} + T_{ji}}{2}\right)[g_i(x_i(t)) - g_i(x_i^*)][g_j(x_j(t)) - g_j(x_j^*)]$$

It follows from Lemma 2 that $V(t) \leq V(0)e_{-\eta}(t,0)$, where $t \in \mathbb{T}_0$. From the definition of $V(t)$, it follows that

$$V(0) \leq \sum_{i=1}^{n} k_i[x_i(0) - x_i^*]^2 \leq \| x(0) - x^* \|^2 \sum_{i=1}^{n} k_i$$

and

$$V(t) \geq \int_{x_i^*}^{x_i(t)}[g_i(\theta) - g_i(x_i^*)]\tilde{\Delta}\theta \geq \frac{1}{2k_i}[g_i(x_i(t)) - g_i(x_i^*)]^2, \qquad (i \in \mathcal{N})$$

Hence

$$[g_i(x_i(t)) - g_i(x_i^*)] \leq \| x(0) - x^* \| \cdot \sqrt{2k_i \sum_{r=1}^{n} k_r} \cdot [e_{-\eta}(t,0)]^{\frac{1}{2}} \text{ for all } t \in \mathbb{T}_0.$$

Then by (3.1), we have

$$D^+ | x_i(t) - x_i^* | \leq -d_i | x_i(t) - x_i^* | + \| x(0) - x^* \| \sum_{j=1}^{n} | T_{ij} | \sqrt{2k_j \sum_{r=1}^{n} k_r} \cdot [e_{-\eta}(t,0)]^{\frac{1}{2}}$$

Again by the Lemma 2, we have

$$| x_i(t) - x_i^* | \leq \| x_i(0) - x_i^* \| e_{-d_i}(t,0) + \| x(0) - x^* \| \sum_{j=1}^{n} | T_{ij} | \sqrt{2k_j \sum_{r=1}^{n} k_r} \cdot \int_0^t [e_{-\eta}(t,0)]^{\frac{1}{2}} e_{-d_i}(t, \sigma(s))\Delta s$$

for all $t \in \mathbb{T}_0$. From Lemma 4, we know that $[e_{-\eta}(t,0)]^{\frac{1}{2}} = e_{-\xi}(t,0)$, where $0 < \xi < \eta < d_i$ $(i \in \mathcal{N})$. Then we have the following

$$\int_0^t [e_{-\eta}(t,0)]^{\frac{1}{2}} e_{-d_i}(t, \sigma(s))\Delta s = \frac{e_{-d_i}(t,0)}{d_i - \xi}[e_{(-\xi)\odot(-d_i)}(t,0) - 1]$$
$$\leq \frac{e_{-d_i}(t,0)e_{(-\xi)\odot(-d_i)}(t,0)}{d_i - \xi} = \frac{e_{-\xi}(t,0)}{d_i - \xi}$$

Together with $e_{-d_i}(t,0) \leq e_{-\xi}(t,0)$, we have

$$| x_i(t) - x_i^* | \leq \| x(0) - x^* \| \left[1 + \frac{1}{d_i - \xi}\sum_{j=1}^{n} | T_{ij} | \sqrt{2k_j \sum_{r=1}^{n} k_r}\right] e_{-\xi}(t,0) = M_i \| x(0) - x^* \| e_{-\xi}(t,0).$$

Let $M = \sum_{i=1}^{n} M_i$, where $M_i = 1 + \frac{1}{d_i - \xi}\sum_{j=1}^{n} | T_{ij} | \sqrt{2k_j \sum_{r=1}^{n} k_r}$. So we have

$$\sum_{i=1}^{n} | x_i(t) - x_i^* | \le M \| x(0) - x^* \| e_{-\xi}(t,0) .$$

This completes the proof.

4 EXAMPLE

We will present a three-dimensional neural network on time scale $\mathbb{T}$, the form as follows:

$$\begin{cases} x_1^{\Delta}(t) = -\frac{2}{3} x_1(t) + T_{11} f(x_1(t)) + T_{12} f(x_2(t)) + T_{13} f(x_3(t)) + I_1 \\ x_2^{\Delta}(t) = -\frac{3}{4} x_2(t) + T_{21} f(x_1(t)) + T_{22} f(x_2(t)) + T_{23} f(x_3(t)) + I_2 \\ x_3^{\Delta}(t) = -\frac{4}{5} x_3(t) + T_{31} f(x_1(t)) + T_{32} f(x_2(t)) + T_{33} f(x_3(t)) + I_3 \end{cases} \tag{5.1}$$

where $t \in \mathbb{T} = P_{1,1}$, $P_{1,1} = \bigcup_{k=0}^{\infty} [2k, 2k+1]$, T_{ij} $(i, j = 1,2)$ and I_i $(i = 1,2)$ are constants and

$f(s) = \frac{1}{2}\left[| s+1 | - | s-1 | \right]$. Choose $\eta = \frac{1}{2}$, we assume that

$T_{11} = -\frac{1}{2}$, $T_{12} = -\frac{1}{3}$, $T_{13} = -\frac{1}{3}$, $T_{21} = -\frac{1}{3}$, $T_{22} = -\frac{3}{2}$, $T_{23} = -\frac{1}{3}$, $T_{31} = -\frac{1}{3}$, $T_{32} = -\frac{1}{3}$, $T_{33} = -\frac{3}{2}$.

Then, matrix $T_{1/2}$ is negative semidefinite. The initial conditions associated with (5.1) are of the forms $(x_1(0), x_2(0), x_3(0))^T = (0.3, -1.2, -3.2)$, $t \in [0,1]_{\mathbb{T}}$. By Theorem 1, the neural network has a unique equilibrium which is globally exponentially stable. We can refer to Figure 1.

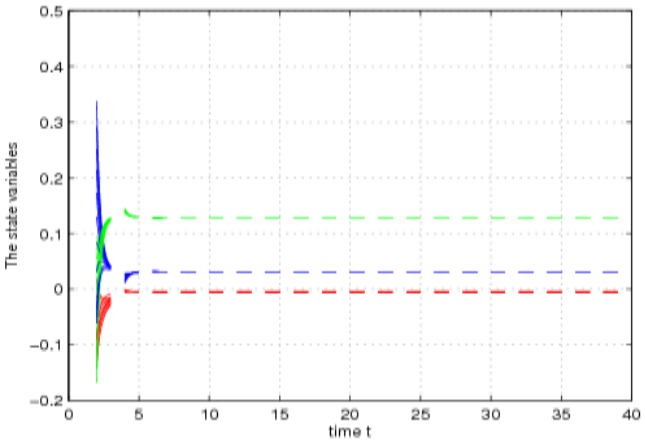

FIG. 1 EXPONENTIAL CONVERGENCE OF (5.1)

5 CONCLUSIONS

By applying the time scale calculus theory and the Lyapunov function method, we obtain some results to ensure exponential convergence of a class of neural networks on time scales. The stability criteria can be easily checked in practice by simple algebraic method and are applicable for continuous and discrete network systems.

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