Mathematical Computation
June 2015, Volume 4, Issue 2, PP.50-55

Modified Filled Function to Solve Nonlinear Programming Problem

Mengxiang Li, Youlin Shang[#], Guanlin Wang

School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang, 471023, China

[#] Corresponding author, Email: mathshang@sina.com

Abstract

Filled function method is an approach to find the global minimum of nonlinear functions. Many Problems, such as computing, communication control, and management, in real applications naturally result in global optimization formulations in a form of nonlinear global integer programming. This paper gives a modified filled function method to solve the nonlinear global integer programming problem. The properties of the proposed modified filled function are also discussed in this paper. The results of preliminary numerical experiments are also reported.

Keywords: Filled Function, Global Optimization, Local Minimizer, Communication Control, Nonlinear Integer Programming

1 INTRODUCTION

Communication Control, management and decision making problems such as capital budgeting, production planning, capacity planning, reliability networks and chemical engineering process, in real applications naturally result in global optimization formulations in a form of nonlinear global integer programming. For example, a plant upgrade problem, a heat exchange network optimization problem and a loading and dispatching problem in a random flexible manufacturing system both aim at searching the global minima of the objective function with constrained conditions. The solutions of these problems all involve the following mathematical problem:

$$\text{(DP)} \quad \min f(x), \ \text{s.t.} \ x \in \Omega$$

Many methods have been proposed to solve this problem, including filled function method [1-8], tunneling method [11], etc. The filled function and the filled function method was first put forward by Ge's paper [3], and many other filled functions have been put forward afterwards [4-8]. The idea of this method is to construct a filled function P(X) and by minimizing P(X) to escape from a given local minimizer x_1^* of the original objective function $f(x)$.

With regard to discrete nonlinear programming problem, the approaches of continuity are presented by Ge's paper [9], and Zhang's paper [10]. This paper proposes a modified filled function to solve the discrete nonlinear programming problem.

The paper is organized as follows: Section 2 presents a modified filled function and discusses its properties. Next, in section 3, a modified filled function algorithm is presented and the results of preliminary numerical experiments are reported. Finally, conclusions are included in section 4.

2 MODIFIED FILLED FUNCTION AND ITS PROPERTIES

In this section, we propose a modified filled function of $f(x)$ at a local minimizer x_1^* and discuss its properties. Let x_1^* be the current local minimizer of problem (DP) and integer set

$$S_1 = \{x \in \Omega : f(x) \geq f(x_1^*)\}, \quad S_2 = \Omega \setminus S_1$$

For the unconstrained continuous global optimization problem

$$\text{(CP)} \quad \min f(x), \ \text{s.t.} \ x \in R^n$$

Where $f(x)$ is a (twice) continuously differentiable function on R^n and assume that it is globally convex function, from the globally convex property of $f(x)$, we known that there exists a closed and bounded domain Ω contains all the global minimizers of $f(x)$. Paper [1] gives a definition of the globally convexized filled function of $f(x)$ at its local minimizer x_1^* as follows

Definition 2.1[1] A continuous function $U(x)$ is called a globally convexized filled function of $f(x)$ at its minimizer x_1^* if $U(x)$ has the following properties:

(i) $U(x)$ has no stationary point in the region $S_1 = \{x \in \Omega : f(x) \geq f(x_1^*)\}$ except a prefixed point $x_0 \in S_1$ that is a minimizer of $U(x)$.

(ii) $U(x)$ does have a minimizer in the region S_2.

(iii) $U(x) \to +\infty$ as $x \to +\infty$.

With regard to the nonlinear integer programming problem (DP) in section 1 of this paper, a definition of the filled function of f (x) at its local minimizer x_1^* on the basis of Definition 2.1[1] is given in [8] as follows

Definition 2.2[8] Function $U(x)$ is called a filled function of $f(x)$ at its minimizer x_1^* for nonlinear integer programming problem if $U(x)$ has the following properties:

(i) $U(x)$ has no minimizer in the set $S_1 = \{x \in \Omega : f(x) \geq f(x_1^*)\}$ except a prefixed point $x_0 \in S_1$ that is a minimizer of $U(x)$.

(ii) If $S_2 \neq \phi$, then $U(x)$ does have a minimizer in the set S_2.

This paper gives a modified definition of the filled function of f (x) at its local minimizer x_1^* for nonlinear integer programming problem (DP) on the basis of Definition 2.1[1] and Definition 2.2[8] as follows:

Definition 2.3 $P(x)$ is called a filled function of $f(x)$ at its minimizer x_1^* for nonlinear integer programming problem if $P(x)$ has the following properties:

(i) $P(x)$ has no local minimizer in the set $S_1 \setminus x_0 = \{x \in \Omega \setminus x_0 : f(x) \geq f(x_1^*)\}$. Prefixed point x_0 in the set S_1 and is not necessarily a local minimizer of $P(x)$.

(ii) If x_1^* is not a global minimizer of $f(x)$, then there exists a local minimizer of $P(x)$, such that x_i^k, that is $x_1 \in S_2$.

Definition 2.3 is different from **Definition 2.1** and **Definition 2.2**. It is based on the discrete set in the Euclidean space and x_0 is not necessarily local minimizer of $P(x)$.

Therefore, we can present a modified filled function of $f(x)$ at its local minimizer x_1^* is as follows:

$$P(x) = \eta(\|x - x_0\|) - A \cdot [\min\{f(x) - f(x_1^*), 0\}]^2, \tag{2.1}$$

Where $A > 0$ is a parameter, prefixed point $x_0 \in \Omega$ satisfies the condition $f(x_0) \geq f(x_1^*)$ and function η(t) need to satisfy the following conditions:

(i) $\eta(t)$ is strictly monotone increasing function for any $t \in [0, +\infty)$;

(ii) $\eta(0) = 0$.

Lemma 2.1[2] For any integer point $x_0 \in \Omega$, if $x \neq x_0$, there exists a $d \in D = \{\pm e_i : i = 1,2,\cdots,n\}$ such that

$$\|x + d - x_0\| < \|x - x_0\|, \tag{2.2}$$

Theorem 2.1 $P(x)$ has no local minimizer in the set $S_1 \setminus x_0 = \{x \in \Omega \setminus x_0 : f(x) \geq f(x_1^*)\}$ for any $A > 0$.

Proof: From lemma 2.1, we know that, for any $x \in S_1$ and $x \neq x_0$, there exists a $d \in D$ such that

$$\|x + d - x_0\| < \|x - x_0\|.$$

Consider the following two cases:

(1) If $f(x_1^*) \le f(x+d) \le f(x)$, or $f(x_1^*) \le f(x) \le f(x+d)$, then

$$P(x+d) = \eta(\|x+d-x_0\|) - A[\min\{f(x+d) - f(x_1^*), 0\}]^2 = \eta(\|x+d-x_0\|) < \eta(\|x-x_0\|)$$

$$= \eta(\|x-x_0\|) - A[\min\{f(x) - f(x_1^*), 0\}]^2 = P(x)$$

Therefore, x is not a local minimizer of function $P(x)$.

(2) If $f(x+d) \le f(x_1^*) \le f(x)$, then

$$P(x+d) = \eta(\|x+d-x_0\|) - A[\min\{f(x+d) - f(x_1^*), 0\}]^2 = \eta(\|x+d-x_0\|) - A[f(x+d) - f(x_1^*)]^2$$

$$\le \eta(\|x+d-x_0\|) < \eta(\|x-x_0\|) = \eta(\|x-x_0\|) - A[\min\{f(x) - f(x_1^*), 0]^2 = P(x)$$

Therefore, it is also show that x is not a local minimizer of function $P(x)$.

From **Theorem 2.1**, we know that the function $P(x)$ satisfies the first property of **Definition 2.3** without any further assumption on the parameter $A > 0$.

A question arises how large the parameter A should be such that $P(x)$ has a local minimizer in the set S_2. To answer this question, we have the following Theorem.

Theorem 2.2 Let $S_2 \ne \phi$. If the parameter $A > 0$ satisfies the condition $A > C / [f(x^*) - f(x_1^*)]^2$, where $C \ge \max_{x \in \Omega} \eta(\|x - x_0\|)$, x^* is a global minimizer of $f(x)$, then $P(x)$ has a local minimizer in the set S_2.

Proof: Since the set S_2 is nonempty and x^* is a global minimizer of $f(x)$, $f(x^*) < f(x_1^*)$ holds and

$$P(x^*) = \eta(\|x^* - x_0\|) - A \cdot [\min\{f(x^*) - f(x_1^*), 0\}]^2 = \eta(\|x^* - x_0\|) - A \cdot [f(x^*) - f(x_1^*)]^2$$

$$\le C - A[f(x^*) - f(x_1^*)]^2$$

When $A > 0$ and satisfies the given condition, we have $P(x^*) < 0$.

On the other hand, for any $y \in S_1$, we have

$$P(y) = \eta(\|y - x_0\|) - A \cdot [\min\{f(y) - f(x_1^*), 0\}]^2 = \eta(\|y - x_0\|) \ge 0.$$

Therefore, the global minimizer of $P(x)$ belong to the set S_2, that is, function $P(x)$ has a local minimizer in the set S_2.

Theorem 2.3 Suppose that ε is a small positive constant and $A > C / \varepsilon^2$, then for any x_1^* of $f(x)$ such that $f(x_1^*) \ge f(x^*) + \varepsilon$, $P(x)$ has a local minimizer in the set S_2, where x^* is a global minimizer of $f(x)$.

Proof: Since $f(x_1^*) - f(x^*) \ge \varepsilon$, we have $\{C / [f(x^*) - f(x_1^*)]^2\} \le \{C / \varepsilon^2\}$. It follows from **Theorem 2.2** that the conclusions of this Theorem hold.

We construct the following auxiliary nonlinear integer programming problem (ADP) relate to the problem (DP):

$$\text{(ADP)} \ \min P(x), \text{ s.t. } x \in \Omega$$

3 MODIFIED FILLED FUNCTION ALGORITHM AND NUMERICAL RESULTS

In this section, we put our modified filled function in the following algorithm to solve the problem (DP).

The local minimizer of $f(x)$ over Ω is obtained by the following Algorithm.

Algorithm 1[7]

Step 1. Choose any integer $x_0 \in \Omega$.

Step 2. If x_0 is a local minimizer of $f(x)$ over Ω, then stop; otherwise search the neighborhood $N(x_0)$ and obtain a point $x \in N(x_0) \cap \Omega$ such that $f(x) < f(x_0)$.

Step 3. Let $x_0 := x$, go to Step 2.

Algorithm 2 (The modified filled function method)

Step 1. Choose:

(a) choose functions $\eta(t)$ satisfy the conditions in section 2 of this paper;

(b) choose a constant $N_L > 0$ as the tolerance parameter for terminating the minimization process of problem (DP);

(c) choose a small constant ε as a desired optimality tolerance.

Step 2. Input:

(a) input an integer point $x_0 \in \Omega$;

(b) input a constant A satisfying the condition $A > C/[f(x^*) - f(x_1^*)]^2$ or $A > C/\varepsilon^2$.

Step 3. Starting from the point x_0, obtain a local minimizer x_1^* of $f(x)$ over Ω.

(a) if x_0 is a local minimizer of $f(x)$ over Ω, let $x_1^* = x_0$ and go to Step 4;

(b) if x_0 is not a local minimizer of $f(x)$ over Ω, search the neighborhood $N(x_0)$ and obtain a point obtain a point $x \in N(x_0) \cap \Omega$ such that $f(x) < f(x_0)$;

(c) let $x_0 = x$ and go to (a) of Step 3.

Step 4. let $\eta(t) = t$, we construct the filled function $P(x)$ as follows:

$$P(x) = \eta(\|x - x_0\|) - A \cdot [\min\{f(x) - f(x_1^*), 0\}]^2$$

Step 5. Let $N = 0$.

Step 6. If L $N > N_L$, then go to Step 11.

Step 7. Set $N = N + 1$. Choose an initial point on the set Ω. Starting from this point, minimize $P(x)$ on the set Ω using any local minimization method. Suppose that x' is an obtained local minimizer.

Step 8. If $x' = x_0$, go to Step 6; otherwise, go to Step 9.

Step 9. Minimize $f(x)$ on the set Ω from the initial point x', and obtain a local minimizer * x_2^* of $f(x)$.

Step 10. Let $x_1^* = x_2^*$ and go to Step 4.

Step 11. Out put x_1^* of $f(x_1^*)$ as a approximate global minimal solution and global minimal value of problem (DP) respectively.

Example 1 (in [6] and [8])

$$\min f(x) = (x_1 - 1)^2 + (x_n - 1)^2 + n\sum_{i=1}^{n-1}(n-i)(x_i^2 - x_{i+1})^2 \qquad s.t. |x_i| \le 5,\ x_i \text{ is integer, } i = 1, 2, \cdots, n$$

This problem has 11^n feasible points and many local minimizers (4, 6, 7, 10 and 12 local minimizers for n=2, 3, 4, 5 and 6, respectively), but only one global minimum solution: $x_g^* = (1, 1, \cdots, 1)$ with $f(x_g^*) = 0$, for all n. We considered two sizes of the problem: n=2 and 5.

Example 2 (in [6] and [7])

$$\min f(x) = \sum_{i=1}^{n-1}[100(x_{i+1} - x_i^2)^2 + (1 - x_i)^2], \qquad s.t. |x_i| \le 5,\ x_i \text{ is integer, } i = 1, 2, \cdots, n$$

This problem has 11^n feasible points and many local minimizers (5, 6, 7, 9 and 11 local minimizers for n=2, 3, 4, 5 and 6, respectively), but only one global minimum solution: $x_g^* = (1, 1, \cdots, 1)$ with $f(x_g^*) = 0$, for all n. We considered two sizes of the problem: n=5 and 6.

In the following, computational results of some test problems using the above algorithm are summarized. The computer is equipped with Windows XP system with 900 *Z MH CPU*. The symbols are used in the tables are noticed as follows:

n : The number of variables;

T_S : The number of initial points to be chosen;

k : The number of times that the local minimization process of the problem (DP);

x_i^k : The initial point for the k-th local minimization process of problem (DP);

x_f^k : The minimizer for the k-th local minimization process of problem (DP);

x_p^k : The minimizer for the k-th local minimization process of problem (ADP);

QIN : The iteration number for the k-th local minimization process of problem (ADP)

TABLE1. RESULTS OF NUMERICALE EXAMPLE 1, N=5, $A = 283$

T_S	k	x_i^k	x_f^k	$f(x_f^k)$	x_p^k	$f(x_p^k:)$	*QIN*
1	1	(-1,3,-4,3,2)	(0,0,0,0,0)	2	(1,1,1,1,1)	0	12
	2	(1,1,1,1,1)	(1,1,1,1,1)	0			$\geq 10^5+1$
2	1	(2,-2,1,0,0)	(0,0,0,0,0)	2	(1,1,1,1,1)	0	21
	2	(1,1,1,1,1)	(1,1,1,1,1)	0			$\geq 10^5+1$
3	1	(-2,2,0,1,1)	(-1,1,1,1,1)	4	(0,0,0,0,0)	2	4
	2	(0,0,0,0,0)	(0,0,0,0,0)	2	(1,1,1,1,1)	0	8
	3	(1,1,1,1,1)	(1,1,1,1,1)	0			$\geq 10^3+1$

TABLE2. RESULTS OF NUMERICALE EXAMPLE 2, N=5, $A = 448$

T_S	k	x_i^k	x_f^k	$f(x_f^k)$	x_p^k	$f(x_p^k:)$	*QIN*
1	1	(-2,-3,-1,-4,5)	(0,0,0,-2,4)	412	(0,0,0,0,0)	4	5
	2	(0,0,0,0,0)	(1,1,1,1,1)	0	(1,1,1,1,1)	0	41
	3	(1,1,1,1,1)	(1,1,1,1,1)	0			$\geq 10^5+1$
2	1	(-4,-2,-3,-1,5)	(0,0,0,-2,4)	412	(1,1,1,1,1)	0	0
	2	(1,1,1,1,1)	(1,1,1,1,1)	0			$\geq 10^5+1$
3	1	(0,0,-2,0,0)	(0,0,0,0,0)	4	(1,1,1,1,1)	0	46
	2	(1,1,1,1,1)	(1,1,1,1,1)	0			$\geq 10^5+1$
4	1	(-4,-2,-3,-1,5)	(0,0,0,-2,4)	412	(1,1,1,2,4)	101	0
	2	(0,0,0,0,0)	(1,1,1,1,1)	0	(1,1,1,1,1)	0	12
	3	(1,1,1,1,1)	(1,1,1,1,1)	0			$\geq 10^5+1$

4 CONCLUSIONS

This paper gives a modified filled function method to solve the nonlinear global integer programming problems, such as computing, communication control, and management, etc. The properties of the proposed modified filled function are also discussed in this paper. The results of preliminary numerical experiments are also reported of the proposed method.

ACKNOWLEDGMENT

This work was supported by National Natural Science Foundation of China (No. 11471102) and Natural Science Foundation of Henan Province (N0. 094300510050).

REFERENCES

[1] Lucid, S., Piccialli, V.: New Classes of Globally Convexized Filled Functions for Global Optimization. J. Global Optimiz. 24, 219–236 (2002)

[2] Ge, R.P., Qin, Y.F.: The Global Convexized Filled Functions for Globally Optimization. Applied Mathematics and Computation 35, 131–158 (1990)
[3] Ge, R.P.: A Filled Function Method for Finding a Global Minimizer of a Function of Several Variables. Mathematical Programming 46, 191–204 (1990)
[4] Shang, Y.L., Zhang, L.S.: A Filled Function Method for Finding a Global Minimizer on Global Integer Optimization. J. Computat. Appl. Math. 181, 200–210 (2005)
[5] Shang, Y.L., Zhang, L.S.: Finding Discrete Global Minimizer with a Filled Function for Integer Programming. Europ. J. Operat. Res. 189, 31–40 (2008)
[6] Shang, Y.L., Pu, D.G., Jiang, A.P.: Finding Global Minimizer with One-parameter Filled Function on Unconstrained Global Optimization. Appl. Math. Comput. 191, 176–182 (2007)
[7] Shang, Y.L., Han, B.S.: One-parameter Quasi-filled Function Algorithm for Nonlinear Integer Programming. J. Zhejiang Univers. SCIENCE 6A, 305–310 (2005)
[8] Zhu, W.X.: A Filled Function Method for Nonlinear Integer Programming. Chinese ACTA of Mathematicae Applicatae Sinica 23, 481–487 (2000)
[9] Ge, R.P., Huang, H.: A Continuous Approach to Nonlinear Integer Programming. Appl. Math. Comput. 34, 39–60 (1989)
[10] Zhang, L.S., Gao, F., Yao, Y.R.: Continuity Methods for Nonlinear Integer Programming. OR Transactions 2, 59–66 (1998)
[11] Levy, A.V., Montalvo, A.: The Tunneling Algorithm for the Global Minimization of Function. SIAM J. Science Statistical Comput. 6(1), 15–29 (1985)

AUTHORS



[1]***Youlin Shang*** was born in Luoyang, China, in 1963. He received his Doctor of Science degree from the Department of Mathematics, Shanghai University in July 2005, afterwards he carried out postdoctoral research in the Department of Applied Mathematics, Tongji University, for 2 years. Professor Shang served in the Department of Applied Mathematics, Henan University of Science and Technology as an assistant professor from 1984 to 1992, a Lecturer from 1992 to 1997, an Associate Professor from 1997 to 2006, and became a full Professor in 2006. Professor Shang's current research interests include nonlinear programming and global optimization.



[1]***Mengxiang Li*** was born in Luoyang, China, in 1990.He was graduated from Shangqiu Normal University in July 2013.He studies in Department of Mathematics and Statistics, Henan University of Science and Technology. His main research interests are in the field of nonlinear programming and global optimization.



[1]***Guanlin Wang*** was born in Peng lai, China, in 1992. He was graduated from Henan University of Science and Technology in July 2014.He studies in Department of Mathematics and Statistics, Henan University of Science and Technology from September 2014. His main research interests are in the field of nonlinear programming and global optimization.