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Three Modified Efficient Iterative Methods for Non-linear Equations

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Abstract

In this paper, we present three modified iterative methods for solving non-linear equations. Firstly, we give a fourth-order convergent iterative method. Then based on the fourth-order method we propose two modified three step Newton-type iterative methods with order of convergence six for solving non-linear equations. All of the methods are free from second derivatives. The efficiency index of Algorithm 1 is equal to that of classical Newton's method 1.414, while the efficiency index of Algorithm 2 and Algorithm 3 is 1.431 which is better than that of classical Newton's method. Several numerical results demonstrate that the proposed methods are more efficient and perform better than classical Newton's method and some other methods.

Keywords: Iterative Method; Non-linear Equation; Order of Convergence; Newton's Method; Efficiency Index

1 INTRODUCTION

We consider iterative methods to find a simple root x^* of a non-linear equation

$$f(x)=0, \tag{1}$$

where $f: D \subseteq R \to R$ for an open interval D is a scalar function, and it is sufficiently smooth in a neighbourhood of x^*.

Classical Newton's method (NM for simplicity) is a basic and important method for solving non-linear equation (1) by the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2}$$

which is quadratically convergent in the neighbourhood of x^* [7].

In past decades, much attention has been paid to develop iterative methods for solving nonlinear equations and many iterative methods have been developed [2-18].

Motivated and inspired by the ongoing activities in this direction, in this paper, we present three iterative methods. All of the methods are free from second derivatives. Algorithm 1 is fourth-order convergent, and requires two evaluations of the functions and two evaluations of derivative in each step. Algorithm 2 and 3 are sixth-order convergent, and they require three evaluations of the functions and two evaluations of derivative per iteration. Some numerical results are given to illustrate the efficiency of the methods.

2 THREE MODIFIED METHODS AND THEIR CONVERGENCE ANALYSIS

Firstly, let us consider the following iterative method.

ALGORITHM 1. For given x_0, we consider the modified Newton-type iteration scheme

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{3}$$

$$x_{n+1} = x_n - \frac{3f'(x_n) - f'(y_n)}{f'(x_n) + f'(y_n)} \frac{f(y_n)}{f'(x_n)}, \tag{4}$$

For Algorithm 1, we have the following convergence result.

THEOREM 1. Assume that the function $f: D \subseteq R \to R$ has a single root $x^* \in D$, where D is an open interval. If $f(x)$ is sufficiently smooth in the interval D, then Algorithm 1 is fourth-order convergent in a neighborhood of x^* and it satisfies error equation

$$e_{n+1} = 3c_2^3 e_n^4 + O(e_n^5) \tag{5}$$

where

$$e_n = x_n - x^*,\ c_k = \frac{f^{(k)}(x^*)}{k!\, f'(x^*)},\ k = 1, 2, \cdots \tag{6}$$

PROOF. Let x^* be the simple root of $f(x)$, $e_n = x_n - x^*$, $d_n = y_n - x^*$, where

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

and

$$c_k = \frac{f^{(k)}(x^*)}{k!\, f'(x^*)},\ k = 1, 2, \cdots$$

Using Taylor expansion and taking into account $f(x^*) = 0$, we have

$$f(x_n) = f'(x^*)\left[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + O(e_n^6)\right]. \tag{7}$$

Furthermore, we have

$$f'(x_n) = f'(x^*)\left[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + O(e_n^5)\right]. \tag{8}$$

Dividing (7) by (8) gives us

$$\begin{aligned} \frac{f(x_n)}{f'(x_n)} &= e_n - c_2 e_n^2 + \left(2c_2^2 - 2c_3\right) e_n^3 + \left(7c_2 c_3 - 4c_2^3 - 3c_4\right) e_n^4 \\ &\quad + \left(8c_2^4 + 6c_3^2 - 20c_2^2 c_3 + 10c_2 c_4 - 4c_5\right) e_n^5 + O\left(e_n^6\right), \end{aligned} \tag{9}$$

so we have

$$\begin{aligned} d_n = e_n - \frac{f(x_n)}{f'(x_n)} &= c_2 e_n^2 - \left(2c_2^2 - 2c_3\right) e_n^3 - \left(7c_2 c_3 - 4c_2^3 - 3c_4\right) e_n^4 \\ &\quad - \left(8c_2^4 + 6c_3^2 - 20c_2^2 c_3 + 10c_2 c_4 - 4c_5\right) e_n^5 + O\left(e_n^6\right), \end{aligned} \tag{10}$$

$$f(y_n) = f'\left(x^*\right)\left[d_n + c_2 d_n^2 + c_3 d_n^3 + c_4 d_n^4 + c_5 d_n^5 + O\left(d_n^6\right)\right]. \tag{11}$$

Substituting (10) into (11), we obtain

$$\begin{aligned} f(y_n) = f'\left(x^*\right)[&c_2 e_n^2 - \left(2c_2^2 - 2c_3\right) e_n^3 - \left(7c_2 c_3 - 5c_2^3 - 3c_4\right) e_n^4 \\ &- \left(12c_2^4 + 6c_3^2 - 24c_2^2 c_3 + 10c_2 c_4 - 4c_5\right) e_n^5 + O\left(e_n^6\right)]. \end{aligned} \tag{12}$$

Therefore, dividing (12) by (8), we have

$$\begin{aligned} \frac{f(y_n)}{f'(x_n)} &= c_2 e_n^2 + \left(2c_3 - 4c_2^2\right) e_n^3 + \left(13c_2^3 - 14c_2 c_3 + 3c_4\right) e_n^4 \\ &\quad - \left(38c_2^4 + 12c_3^2 - 64c_2^2 c_3 + 20c_2 c_4 - 4c_5\right) e_n^5 + O\left(e_n^6\right). \end{aligned} \tag{13}$$

It follows from (12) that

$$\begin{aligned} f'(y_n) = f'\left(x^*\right)[&2c_2 e_n - \left(6c_2^2 - 6c_3\right) e_n^2 - \left(28c_2 c_3 - 20c_2^3 - 12c_4\right) e_n^3 \\ &- \left(60c_2^4 + 30c_3^2 - 120c_2^2 c_3 + 50c_2 c_4 - 20c_5\right) e_n^4 + O\left(e_n^5\right)]. \end{aligned} \tag{14}$$

Thus, substituting (8), (14) into (4), we have

$$e_{n+1} = 3c_2^3 e_n^4 + O(e_n^5). \tag{15}$$

This means that Algorithm1 is fourth-order convergent. The proof is completed.

Next, we give further modification of Algorithm1, and obtain the following modified newton-type three step iterative method.

ALGORITHM 2. For given x_0 , we consider three step Newton-type iteration scheme for solving non-linear equation (1) by the iterative scheme

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{16}$$

$$z_n = x_n - \frac{3f'(x_n) - f'(y_n)}{f'(x_n) + f'(y_n)} \frac{f(y_n)}{f'(x_n)}, \tag{17}$$

$$x_{n+1} = z_n + \frac{3f'(x_n) + f'(y_n)}{f'(x_n) - 5f'(y_n)} \frac{f(z_n)}{f'(x_n)}. \tag{18}$$

For Algorithm 2, we have the following convergence result.

THEOREM 2.Assume that the function $f : D \subseteq R \to R$ has a single root $x^* \in D$, where D is an open interval. If $f(x)$ is sufficiently smooth in the interval D , then Algorithm 2 is sixth-order convergent in a neighborhood of x^* and it satisfies error equation

$$e_{n+1} = 3c_2^5 e_n^6 + O(e_n^7) \tag{19}$$

where

$$e_n = x_n - x^*,\ c_k = \frac{f^{(k)}(x^*)}{k!\, f'(x^*)},\ k = 1, 2, \cdots$$

PROOF. Let x^* be the simple root of $f(x)$,

$$c_k = \frac{f^{(k)}(x^*)}{k!\, f'(x^*)},\ k = 1, 2, \cdots$$

and

$$e_n = x_n - x^*.$$

Consider the iteration function $F(x)$ defined by

$$F(x) = z(x) + \frac{3f'(x) + f'(y(x))}{f'(x) - 5f'(y(x))} \frac{f(z(x))}{f'(x)} \tag{20}$$

where

$$z(x) = x - \frac{3f'(x) - f'(y(x))}{f'(x) + f'(y(x))} \frac{f(y(x))}{f'(x)},$$

$$y(x) = x - \frac{f(x)}{f'(x)}.$$

By some computations using Maple we have

$$F(x^*) = x^*,\ F^{(i)}(x^*) = 0,\ i = 1, 2, 3, 4, 5,$$

$$F^{(6)}(x^*) = \frac{135 f''(x^*)^5}{2 f'(x^*)^5}. \tag{21}$$

Furthermore, from the Taylor expansion of $F(x_n)$ around x^*, we obtain

$$x_{n+1} = F(x_n) = F(x^*) + F'(x^*)(x_n - x^*) + \frac{F^{(2)}(x^*)}{2!}(x_n - x^*)^2 + \frac{F^{(3)}(x^*)}{3!}(x_n - x^*)^3$$

$$+ \frac{F^{(4)}(x^*)}{4!}(x_n - x^*)^4 + \frac{F^{(5)}(x^*)}{5!}(x_n - x^*)^5 + \frac{F^{(6)}(x^*)}{6!}(x_n - x^*)^6 + O((x_n - x^*)^7). \quad (22)$$

Substituting (21) into (22) yields

$$x_{n+1} = x^* + e_{n+1} = x^* + 3c_2^5 e_n^6 + O(e_n^7). \quad (23)$$

Therefore, we have

$$e_{n+1} = 3c_2^5 e_n^6 + O(e_n^7) \quad (24)$$

which means that the order of convergence of Algorithm 2 is six. This completed the proof.

Then, we are in the position to give the other modified newton-type three step iterative method.

ALGORITHM 3. For given x_0, we consider the following three step iteration scheme

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad (25)$$

$$z_n = x_n + \frac{f'(x_n) + f'(y_n)}{f'(x_n) - 3f'(y_n)} \frac{f(y_n)}{f'(x_n)}, \quad (26)$$

$$x_{n+1} = z_n - \frac{3f'(x_n) - f'(y_n)}{f'(x_n) + f'(y_n)} \frac{f(z_n)}{f'(x_n)}. \quad (27)$$

For Algorithm 3, we have the following convergence result.

THEOREM 3. Assume that the function $f : D \subseteq R \to R$ has a single root $x^* \in D$, where D is an open interval. If $f(x)$ is sufficiently smooth in the interval D, then Algorithm 3 is sixth-order convergent in a neighborhood of x^* and it is satisfied with error equation

$$e_{n+1} = -4c_2^5 e_n^6 + O(e_n^7) \quad (28)$$

where

$$e_n = x_n - x^*,\ c_k = \frac{f^{(k)}(x^*)}{k!\, f'(x^*)},\ k = 1, 2, \cdots$$

PROOF. Let x^* be the simple root of $f(x)$,

$c_k = \frac{f^{(k)}(x^*)}{k!\, f'(x^*)}$, $k = 1, 2, \cdots$, and $e_n = x_n - x^*$.

Consider the iteration function $F(x)$ defined by

$$F(x) = z(x) - \frac{3f'(x) - f'(y(x))}{f'(x) + f'(y(x))} \frac{f(z(x))}{f'(x)} \quad (29)$$

where

$$z(x) = x + \frac{f'(x) + f'(y(x))}{f'(x) - 3f'(y(x))} \frac{f(y(x))}{f'(x)},$$

$$y(x) = x - \frac{f(x)}{f'(x)}.$$

By some computations using Maple we have

$F(x^*) = x^*,\ F^{(i)}(x^*) = 0,\ i = 1, 2, 3, 4, 5,$

$$F^{(6)}(x^*) = -\frac{90 f''(x^*)^5}{f'(x^*)^5}. \tag{30}$$

Furthermore, from the Taylor expansion of $F(x_n)$ around x^*, we obtain

$$x_{n+1} = F(x_n) = F(x^*) + F'(x^*)(x_n - x^*) + \frac{F^{(2)}(x^*)}{2!}(x_n - x^*)^2 + \frac{F^{(3)}(x^*)}{3!}(x_n - x^*)^3$$

$$+ \frac{F^{(4)}(x^*)}{4!}(x_n - x^*)^4 + \frac{F^{(5)}(x^*)}{5!}(x_n - x^*)^5 + \frac{F^{(6)}(x^*)}{6!}(x_n - x^*)^6 + O((x_n - x^*)^7). \tag{31}$$

Substituting (30) into (31) yields

$$x_{n+1} = x^* + e_{n+1} = x^* - 4c_2^5 e_n^6 + O(e_n^7). \tag{32}$$

Therefore, we have

$$e_{n+1} = -4c_2^5 e_n^6 + O(e_n^7), \tag{33}$$

which means that the order of convergence of Algorithm 3 is also six. The proof is completed.

Currently, we consider efficiency index defined by $p^{1/\omega}$, where p is the order of the method and ω is the number of function evaluations in each step required by the method. It is easy to see that the efficiency index of Algorithm 1is 1.414. The efficiency index of Algorithm 2 and Algorithm 3 is 1.431 which is better than that of classical Newton's method (NM) 1.414.

3 NUMERICAL RESULTS

Now, we employ Algorithm 1, 2 and 3 presented in this paper to solve some non-linear equations and compare them with NM and the iterative method (PPM for short) Potra and Pták presented in [6]

$$x_{n+1} = x_n - \frac{f(x_n) + f(y_n)}{f'(x_n)} \tag{34}$$

which is cubically convergent with efficiency index $\sqrt[3]{3} \approx 1.442$.

Displayed in TABLE 1 are the number of required iterations (ITs) such that $|f(x_n)| < 1.E-15$.

In TABLE 1, we use the following functions.

$f_1(x) = \cos x - x,\ x^* = 0.73908513321516.$

$f_2(x) = x^3 + 4x^2 - 10,\ x^* = 1.36523001341409.$

$f_3(x) = (x-1)^3 - 1,\ x^* = 2.$

$f_4(x) = e^{x^2+7x-30} - 1,\ x^* = 3.$

$f_5(x) = (x+2)e^x - 1,\ x^* = -0.44285440096708.$

$f_6(x) = x^3 - 2x^2 + x - 1,\ x^* = 1.75488390427225.$

$f_7(x) = \sin^2(x) - x^2 + 1,\ x^* = 1.40449164885154.$

$f_8(x) = \sin x - 0.5,\ x^* = 1.89549426703398.$

TABLE 1. COMPARISON OF ALGORITHM 1, PPM AND NM

Functions	x_0	NM	PPM	Algorithm 1	Algorithm 2	Algorithm 3
f_1	0.4	5	3	3	3	3

TABLE 1. COMPARISON OF ALGORITHM 1, PPM AND NM (CONT.)

Functions	x_0	NM	PPM	Algorithm 1	Algorithm 2	Algorithm 3
f_1	3.1	7	4	4	3	3
f_2	1.2	6	4	3	3	2
	1.4	5	4	3	3	3
f_3	2.4	6	4	3	3	3
	1.6	8	8	7	6	6
f_4	4.2	15	10	8	6	5
	3.4	12	9	7	5	5
f_5	7.9	15	13	10	8	8
	-0.6	5	4	7	5	5
f_6	3.9	8	6	6	4	4
	1.6	5	4	4	3	3
f_7	1.9	5	4	5	3	3
	2.1	6	5	4	4	3
f_8	1.6	5	4	4	3	3
	2	4	3	3	2	2

The computational results in TABLE 1 show that Algorithm 1, 2 and 3 require less ITs than NM and PPM. Therefore, the present methods are of practical interest and can compete with NM and PPM.

4 CONCLUSIONS

In this paper, we propose and analyse a fourth-order convergent iterative method and two modified sixth-order convergent Newton-type iterative methods for solving non-linear equations. The efficiency index of Algorithm 1 is 1.414, while the efficiency index of Algorithm 2 and Algorithm 3 is 1.431 which is better than that of classical Newton's method (NM) 1.414. All of the given methods are free from second derivatives. Some numerical results demonstrate that the proposed methods in this paper perform better than classical Newton's method and PPM as far as the numerical experiments are concerned.

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