

The Periodic Loop-wave Solution and Loop Soliton Solution of the Generalized Vakhnenko Equation

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Abstract

In this paper, the loop wave solution of a generalized Vakhnenko equation has been studied by using planar bifurcation method of dynamic system. As a result, under $\beta = 3$, the representations of periodic loop-wave solution and loop soliton solution are given, as well their planar graphs. These results are useful supplement to investigation of Vakhnenko equation.

Keywords: Generalized Vakhnenko Equation; Traveling Wave Solution; Periodic Loop-wave Solution; Loop Soliton Solution

广义 Vakhnenko 方程的周期圈波和圈孤子解*

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摘要: 本文用动力系统平面分支方法, 研究一个广义 Vakhnenko 方程的圈波. 在 $\beta = 3$ 的参数条件下, 获得了精确的周期圈波和圈孤子解的表达式, 作出了周期圈波和圈孤子的平面图形, 直观的显示了这两种解的动力学性质. 本文的结果丰富了广义 Vakhnenko 方程的研究.

关键词: 广义 Vakhnenko 方程; 行波解; 周期圈波解; 圈孤子解

引言

在文献[1]中, Vakhnenko 首先研究了下列扩张波动方程

$$\frac{\partial}{\partial x} \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right) u + u = 0, \quad (1.1)$$

得到方程(1.1)的周期圈波和圈孤子两类行波解. 此后方程(1.1)称为 Vakhnenko 方程.

在文献[2]中, Matsuno 用时间和空间变量推导出如下方程

$$\alpha u_x - u_{txx} = \beta u_x u_{xx} + u u_{xxx}, \quad (1.2)$$

其中 α 和 β 是常数. 当 $\beta = 2$ 时, 方程(1.2)变为 Harry - Dym 方程^{[3][4]}, 当 $\alpha = 0$, 方程(1.2)变为 Hunter - Saxton 方程^[5]

$$u_x + 3u_x u_{xx} + u_{txx} + u u_{xxx} = 0. \quad (1.3)$$

明显地, $\alpha = -1$ 和 $\beta = 3$ 时, 方程(1.2) 变为(1.3), 因此方程(1.2)又称为广义 Vakhnenko 方程.

文献[6]讨论了 $\beta \in \mathbb{Z}$, $\beta \neq 0, 1$ 时, 方程(1.2)的行波结构, 给出了一些精确行波解. 本文用文献[6-8]中的方法继续研究方程(1.2), 发现方程(1.2)存在周期圈波和圈孤子. 在 $\beta = 3$ 的条件下, 获得了周期圈波和圈孤子解的参数表示, 作出了周期圈波和圈孤子的平面图形.

1 圈波的存在条件

作变换 $\xi = x - ct$, $u(x, t) = \varphi(\xi)$, 其中常数 c 是波速, 在此变换下方程(1.2)中可化为下列常微分方程

$$\alpha\varphi' + c\varphi''' = \beta\varphi'\varphi'' + \varphi\varphi'''. \quad (2.1)$$

对方程(2.1)积分一次, 得行波方程

$$(\varphi - c)\varphi'' = \alpha\varphi - \frac{\beta - 1}{2}(\varphi')^2 + g, \quad (2.2)$$

其中 g 是积分常数。

在方程(2.2)两边同乘 $2(\varphi - c)^{(\beta-2)}\varphi'$ 并积分, 得

$$(\varphi - c)^{(\beta-1)}\left(\frac{d\varphi}{d\xi}\right)^2 = \frac{2\alpha}{\beta}(\varphi - c)^\beta + \frac{2(\alpha c + g)}{\beta - 1}(\varphi - c)^{(\beta-1)} + h, \quad (2.3)$$

其中 h 是积分常数。设

$$d\xi = (\varphi - c)^{\frac{\beta-1}{2}} d\tau, \quad (2.4)$$

则方程(2.3)变为

$$\left(\frac{d\varphi}{d\tau}\right)^2 = \frac{2\alpha}{\beta}(\varphi - c)^\beta + \frac{2(\alpha c + g)}{\beta - 1}(\varphi - c)^{(\beta-1)} + h. \quad (2.5)$$

令

$$f_0(\varphi) = \frac{2\alpha}{\beta}(\varphi - c)^\beta + \frac{2(\alpha c + g)}{\beta - 1}(\varphi - c)^{(\beta-1)}, \quad (2.6)$$

有

$$f'_0(\varphi) = 2(\varphi - c)^{(\beta-2)}(\alpha\varphi + g). \quad (2.7)$$

显然, $\beta > 2$ 时, $f_0(\varphi)$ 有两个驻点 $\varphi_c = c$ 和 $\varphi_\alpha = -\frac{g}{\alpha}$ 。

如果 β 为偶数, 则当 $\alpha < 0$ 时, φ_α 是极大点, 当 $\alpha > 0$ 时, φ_α 是极小点。

如果 β 为奇数, 则当 $\alpha > 0$ 且 $\varphi_c > \varphi_\alpha$ 时, φ_c 是极小点, φ_α 是极大点; 当 $\alpha > 0$ 且 $\varphi_c < \varphi_\alpha$ 时, φ_c 是极大点, φ_α 是极小点; 当 $\alpha < 0$ 且 $\varphi_c > \varphi_\alpha$ 时, φ_c 是极大点, φ_α 是极小点; 当 $\alpha < 0$ 且 $\varphi_c < \varphi_\alpha$ 时, φ_c 是极小点, φ_α 是极大点。

令

$$f(\varphi) = h + f_0(\varphi), \quad (2.8)$$

则(2.3)和(2.5)分别变为

$$\left(\frac{d\varphi}{d\xi}\right)^2 = \frac{f(\varphi)}{(\varphi - c)^{\beta-1}}, \quad (2.9)$$

和

$$\left(\frac{d\varphi}{d\tau}\right)^2 = f(\varphi). \quad (2.10)$$

根据上面的讨论和文献[8]的结论有下列周期圈波的存在定理

定理: 设 $\beta > 2$ 是奇数, 在满足下列两个条件之一时, 方程(1.2)存在周期圈波, 且 h 趋近于 $-f_0(\varphi_\alpha)$ 时, 周期圈波变为圈孤子。

(1) $\alpha > 0$, $\varphi_c < \varphi_\alpha$, $0 < h < -f_0(\varphi_\alpha)$;

(2) $\alpha < 0$, $\varphi_c > \varphi_\alpha$, $0 < h < -f_0(\varphi_\alpha)$.

2 举例

(1) 设 $\beta = 3$, $\alpha > 0$, $\varphi_c < \varphi_\alpha$, $0 < h < -f_0(\varphi_\alpha)$, 则 $f(\varphi)$ 有三个实数根 $\varphi_i (i = 1, 2, 3)$, 满足 $\varphi_1 < \varphi_c < \varphi_2 < \varphi_\alpha < \varphi_3$, 即

$$f(\varphi) = \frac{2\alpha}{3}(\varphi - \varphi_1)(\varphi - \varphi_2)(\varphi - \varphi_3). \quad (3.1)$$

在期间 $[\varphi_1, \varphi_2]$ 内对(2.10)积分, 有

$$\int_{\varphi_1}^{\varphi} \frac{1}{\sqrt{(\varphi_3 - t)(\varphi_2 - t)(t - \varphi_1)}} dt = \int_0^{\xi} \pm \sqrt{\frac{2\alpha}{3}} dt. \quad (3.2)$$

用文献[9]中的积分公式233.00, 得

$$\varphi = \varphi_1 + (\varphi_2 - \varphi_1) \operatorname{sn}^2(w, j), \quad (3.3)$$

其中 $w = \frac{\sqrt{6\alpha(\varphi_3 - \varphi_1)}}{6} \tau$ 是参变量, $j = \sqrt{\frac{\varphi_2 - \varphi_1}{\varphi_3 - \varphi_1}}$ 是雅可比椭圆函数的模。

置(3.3)于(2.4)中积分, 有

$$\int_0^\xi dt = \int_0^w \sqrt{\frac{6}{\alpha(\varphi_3 - \varphi_1)}} [\varphi_1 - c + (\varphi_2 - \varphi_1) \operatorname{sn}^2(w, j)] dt. \quad (3.4)$$

用文献[9]中的积分公式310.02, 得

$$\xi = \sqrt{\frac{6}{\alpha(\varphi_3 - \varphi_1)}} \{(\varphi_3 - c)w - (\varphi_3 - \varphi_1)E[\operatorname{am}(w, j), j]\}. \quad (3.5)$$

于是得到一个周期圈波解

$$\begin{cases} \varphi = \varphi_1 + (\varphi_2 - \varphi_1) \operatorname{sn}^2(w, j) \\ \xi = \sqrt{\frac{6}{\alpha(\varphi_3 - \varphi_1)}} \{(\varphi_3 - c)w - (\varphi_3 - \varphi_1)E[\operatorname{am}(w, j), j]\} \end{cases}. \quad (3.6)$$

h 趋近于 $-f_0(\varphi_\alpha)$ 时, (3.6) 变为

$$\begin{cases} \varphi = \varphi_1 + (\varphi_\alpha - \varphi_1) \tanh^2 w \\ \xi = \sqrt{\frac{6}{\alpha(\varphi_\alpha - \varphi_1)}} [(\varphi_\alpha - c)w - (\varphi_\alpha - \varphi_1) \tanh w] \end{cases} \quad (3.7)$$

是一个圈孤子解。

取 $\alpha = 1, g = -1$, 有 $\varphi_\alpha = 1$ 。再取 $c = -2$, 得 $f_0(1) = -9$ 。再取 $h = 8$, 得 $\varphi_1 = -3.42334, \varphi_2 = 0.378076, \varphi_3 = 1.54526$ 。把相关数据代入(3.6)中, 在 $\xi - u$ 平面上可画出一个周期圈波图, 见图 1(a)。

取 $h = 9$, 则 $\varphi_1 = -3.5, \varphi_\alpha = \varphi_2 = \varphi_3 = 1$ 。把相关数据代入(3.7)中, 在 $\xi - u$ 平面上可画出一个圈孤子图, 见图 2(a)。

(2) 设 $\beta = 3, \alpha < 0, \varphi_c > \varphi_\alpha, 0 < h < -f_0(\varphi_\alpha)$, 则 $f(\varphi)$ 有三个实数根 $\varphi_i (i = 1, 2, 3)$, 满足 $\varphi_1 < \varphi_\alpha < \varphi_2 < \varphi_c < \varphi_3$ 。在期间 $[\varphi_2, \varphi_3]$ 内对(2.10)积分, 有

$$\int_{\varphi}^{\varphi_3} \frac{1}{\sqrt{(\varphi_3 - t)(t - \varphi_2)(t - \varphi_1)}} dt = \int_{\xi}^0 \pm \sqrt{\frac{2\alpha}{3}} dt. \quad (3.8)$$

用文献[9]中的积分公式236.00, 得

$$\varphi = \varphi_3 - (\varphi_3 - \varphi_2) \operatorname{sn}^2(w, j), \quad (3.9)$$

其中 $w = \frac{\sqrt{6\alpha(\varphi_1 - \varphi_3)}}{6} \tau$ 参变量, $j = \sqrt{\frac{\varphi_3 - \varphi_2}{\varphi_3 - \varphi_1}}$ 是雅可比椭圆函数的模。

置(3.9)于(2.4)中积分, 有

$$\int_0^\xi dt = \int_0^w \sqrt{\frac{6}{\alpha(\varphi_3 - \varphi_1)}} [\varphi_3 - c - (\varphi_3 - \varphi_2) \operatorname{sn}^2(w, j)] dt. \quad (3.10)$$

用文献[9]中的积分公式310.02, 得

$$\xi = \sqrt{\frac{6}{\alpha(\varphi_1 - \varphi_3)}} \{(\varphi_1 - c)w + (\varphi_3 - \varphi_1)E[\operatorname{am}(w, j), j]\}. \quad (3.11)$$

于是我们获得另一个周期圈波解

$$\begin{cases} \varphi = \varphi_3 - (\varphi_3 - \varphi_2) \operatorname{sn}^2(w, j) \\ \xi = \sqrt{\frac{6}{\alpha(\varphi_1 - \varphi_3)}} \{(\varphi_1 - c)w + (\varphi_3 - \varphi_1)E[\operatorname{am}(w, j), j]\} \end{cases}. \quad (3.12)$$

h 趋近于 $-f_0(\varphi_\alpha)$ 时, (3.12) 变为

$$\begin{cases} \varphi = \varphi_3 - (\varphi_3 - \varphi_\alpha) \tanh^2 w \\ \xi = \sqrt{\frac{6}{\alpha(\varphi_\alpha - \varphi_3)}} [(\varphi_\alpha - c)w + (\varphi_3 - \varphi_\alpha) \tanh w] \end{cases}. \quad (3.13)$$

取 $\alpha = -1$, $g = -1$, 得 $\varphi_\alpha = -1$ 。再取 $c = 2$, 有 $f_0(-1) = -9$ 。再取 $h = 8$, 则 $\varphi_1 \doteq -1.54526, \varphi_2 \doteq -0.378076, \varphi_3 \doteq 3.423336$, 把相关数据代入(3.12)中, 在 $\xi - u$ 平面上可画出一个周期圈波图, 见图 1(b)。

取 $h = 9$, 则 $\varphi_\alpha = \varphi_1 = \varphi_2 = -1, \varphi_3 = 3.5$. 把相关数据代入(3.13)中, 在 $\xi - u$ 平面上可画出一个圈孤子图, 见图 2(b)。

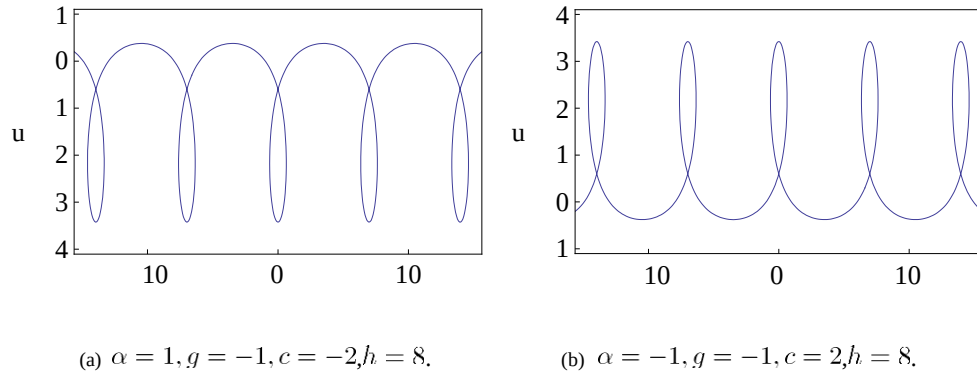


图 1 $\beta = 3$ 时, 方程(1.2)的周期圈波平面图.

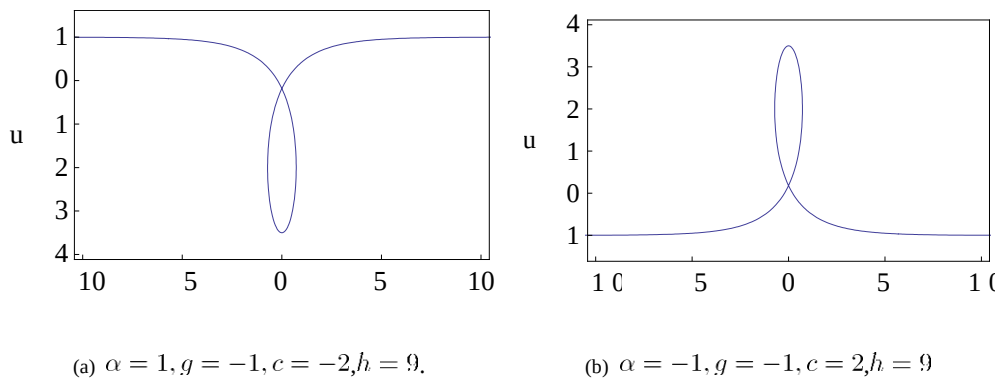


图 2 $\beta = 3$ 时, 方程(1.2)的圈孤子平面图

3 结束语

本文研究了广义 Vakhnenko 方程, 发现此方程在某些特定的参数条件下, 存在周期圈波解, 周期圈波的极限是圈孤子. 取定 $\beta = 3$, 我们计算出了周期圈波和圈孤子的参数表示式. 通过周期圈波解和圈孤子解的参数表达式, 用数学软件画出了周期圈波和圈孤子的平面图形, 见图 1 和图 2 通过平面图能清晰地了解周期圈波和圈孤子的结构. 与现有文献中的结论相比较, 本文对于广义 Vakhnenko 方程的周期圈波的发现是有意义的.

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